Facilitating Productive Discussions

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Capitalize on student thinking to create opportunities to further their mathematical reasoning.

By Nesrin Cengiz

Whole-group classroom discussions about solutions allow teachers to promote reasoning that moves students beyond merely noticing mathematical ideas toward developing a well-connected knowledge of concepts. Creating classroom environments where teachers promote reasoning and engage students in investigating important mathematical phenomena is critical for teaching math for understanding (Ball and Bass 2003; Kline 2008; NCTM 2000; Martin and Kasmer 2009). Nevertheless, many researchers have found that such discussions are challenging for teachers to facilitate in terms of establishing appropriate expectations for participation (Yackel and Cobb 1996), recognizing which aspects of the math to focus on (Ball, Hill, and Bass 2005), and deciding what kind of support to provide for students (Cengiz, Kline, and Grant 2011).

To illustrate the complexity of the work of pursuing student reasoning, read the following vignettes from second-, third-, and fourth-grade classrooms. Then consider the author's suggestions for creating opportunities to promote reasoning.

A set of disparate answers

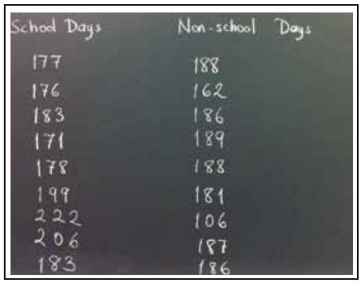
One way to encourage students to make sense of mathematics is to create opportunities for mathematical disagreements and engage students in reasoning about different views (Barlow and McCrory 2011). Mathematics is a discipline that relies on reasoning for validation of ideas; being involved in the process of conflict resolution supports the development of students' reasoning (Wood 1999). The following episode provides an example of how one teacher used disagreement to encourage student thinking. Key components of this discussion are the specific instructional actions she takes: prompting students to consider whether more than one answer could be correct, to identify reasonable solutions, to offer counterexamples, and to compare the efficiency of strategies. Third graders were solving this problem:

> Do we have more school days or nonschool days in a year?

After a brief discussion on what to count as nonschool days (weekends, conference days, etc.), the class labeled school days and nonschool days on calendars. Then the students worked individually and in pairs to total the number of school days and nonschool days. As the teacher listened to students' conversations, she collected all the different answers, listed them on the board (see **fig. 1**), and encouraged students to examine the list. She posed two focusing questions:

FIGURE 1

Students added the number of school days and nonschool days.



1. Could more than one answer be right?
2. Would somebody talk about whether more than one of them could be correct or not?

When students are accustomed to solving problems in more than one way, they sometimes think that a computation problem could have multiple correct answers. The teacher (*T*) used this issue to encourage discussion around a potential disagreement:

Alicia: We solved it differently, so more than one of them could be correct.
Mia: I think there can't be more than one right answer, because we all worked with the same information, um, calendar.
T: So, we have a disagreement here. Could somebody repeat what Alicia and Mia are saying?
Jorge: Alicia is saying that more than one of them could be correct, but Mia is saying that they cannot.
T: Alicia is saying that since we used different ways to calculate the number of school days and nonschool days, we could have different correct answers. Mia is saying that the calendars we marked were all the same, so the answers have to be the same. What do the rest of you think about Alicia's and Mia's reasoning? Do you agree with Alicia or Mia? Why?

When Alicia and Mia shared their thinking, they both supplied justifications, an established norm in this classroom. Whenever students shared an idea or a solution, they were expected to explain why their thinking made sense. They were often asked to repeat their peers' responses, which helped everyone realize the need to listen carefully to one another. The teacher's and students' restatements allowed the whole group to hear different interpretations of the claims and provided all students with access to the shared views.

After a brief discussion, some students agreed with Mia. But at this point, the teacher intentionally refrained from pushing for whole-class agreement. Instead, she encouraged everybody to continue to think about this issue: "Which of these number combinations would make sense?"

One student suggested that the sum of school days and nonschool days should be 365 and that having numbers on the list that did not add up to 365 therefore made no sense. The teacher first restated this claim and then invited the third graders to consider possible counterexamples:

> If we add the school days plus the nonschool days, this should be all the days in a year. Is that true? Are there some other kinds of days that would not fit in [the categories of] school days or nonschool days? Can you think of any kind?

Asking for a counterexample encouraged students to consider possibilities that might contradict the generalization of school days and nonschool days making a whole year. Students realized that there were no other kinds of days, and they used this conclusion to eliminate some of the number combinations (i.e., those that did not add up to 365) from the original list.

The teacher asked which kinds of strategies they used to find the total number of days. The most common strategy was putting together the number of school days and nonschool days from each month or each week. A few pairs counted days one by one. Wondering whether her students could recognize that some strategies were more efficient than others, the teacher asked, "Does it matter if we put the number of school days together in chunks or one by one?" Posing this question created an opportunity for students to compare strategies relative to their efficiency and helped them consider and develop strategies that were more sophisticated. This approach also reinforced the idea that no matter which strategy they used, only one answer for the number of school days and nonschool days is correct.

The exchange above suggests a simple technique that teachers can use to encourage reasoning: *Begin discussions by providing all solutions, correct and incorrect, for students to consider*. Having a variety of solutions creates mathematical conflict and supplies a basis for considering the mathematical context of the problem as well as the reasonableness of the solutions.

Analysis and justification

Current recommendations call for teachers to encourage children to use a variety of strategies, even for paper-and-pencil computation (NCTM 2000). However, what is less clear is how to facilitate productive discussions around those strategies. Reasoning may or may not be the focus of such dialogues, depending on the extent to which teachers work to go beyond students' "descriptions" of their strategies. Therefore, an important aspect of these discussions should be to reflect on whether a solution is logical and valid for the original problem. Consider the following example from a second-grade classroom where students first worked individually and then in pairs on this story problem:

> Yesterday at the park, I counted sixty-nine pigeons. When a big dog walked by, forty-seven of them flew away. How many were still there?

Selected students wrote their solutions on the board, one of which follows:

$69 - 47 = ?$
$60 - 40 = 20, 9 - 7 = 2$
$20 + 2 = 22$

The teacher then asked, "Can you explain your strategy?" and the following discussion ensued.

Ian: I first subtracted forty from sixty, and then seven from nine. Then I added twenty and two and got twenty-two.
T: Does this strategy make sense?
Multiple students: Yes. No.
T: What questions do you have for Ian?

The teacher realized that Ian had only *described* his strategy, so in pursuit of the reasoning behind Ian's solution, she asked his classmates to pose questions. Some of them asked Ian questions that suggested they were actively listening to his explanation:

- "Why did you take forty away from sixty?"
- "Where did the nine and seven come from?"
- "I thought this was a subtraction problem; why are you adding at the end?"

Knowing how common it is for students to erroneously subtract two from twenty at the end of this strategy, the teacher decided to pursue this issue by asking students to provide reasoning for adding two and twenty.

T: So, who can tell me why Ian added there [*pointing at the last step*] rather than subtracted?
Marcus: Because he wouldn't want to get the wrong answer?
T: But he wouldn't know that was the wrong answer.
Mateo: Because it wouldn't make sense?
T: Why wouldn't it make sense? What wouldn't make sense about it? I want you to tell me why he added.
Cristina: Because those are the two answers that he got when he took away.

Notice how one of the students argued against doing a particular step because it would not give the right answer. The teacher recognized that this was not a sufficient explanation for why two had to be added to twenty, and she continued to pursue a more convincing justification. With that said, the second graders were still struggling to articulate an explanation, and the teacher was struggling to recall additional instructional actions that would push the conversation further. This work can be challenging and, at times, frustrating for the teacher and students alike. But without a complete justification, the opportunity is lost for students to develop reasoning that is increasingly sophisticated. Stop and think for a moment about the last step in the procedure, and consider the following:

- Why do you add twenty and two together?
- What would you consider to be a good justification for doing so?
- Would your explanation make sense to a second grader?

These are not trivial questions, and it is difficult to push the discussion any further without having clear responses in mind before engaging with students. Perhaps your responses to the questions have also given you some ideas for continuing this discussion with these students. Here are two more instructional actions, which the teacher in the episode above did not take but which could be beneficial:

- Use the context of the problem.
- Generate diagrams.

The context

With the first action, the conversation might have proceeded in this way:

T: Let's decide as a group if this solution makes sense.
Marcus: It works because we first took forty away from sixty, then seven from nine. Then we added twenty and two.
T: What does taking forty away from sixty mean in terms of the pigeons in the park?
Mia: It's like, first forty of the sixty pigeons flew away, and there are twenty left.
T: Then what?
Mateo: Nine pigeons were sitting on the corner of the park. Then seven of those flew away. Now, two of the pigeons and the other twenty are still at the park. Altogether, there are twenty-two pigeons left.
T: Why are we adding twenty and two again?
Amanda: See, those are all the pigeons that are left in the park. You need to put them together.

Now think about how this exchange is different from the first one, where the context was not used. Do they look different from each other in terms of the reasoning that students provide for the same solution? In the first exchange, despite the teacher's *Why?* questions, students struggled with reasoning about the strategy because the questions did not seem to promote the kind of thinking needed to understand why they added two and twenty. In the second exchange, by specifically asking students to use the context of the problem to supply an interpretation of the solution steps, the teacher enabled her students to construct a foundation for their reasoning.

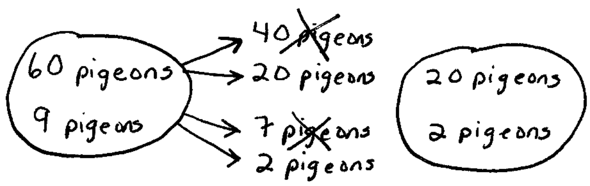
Diagrams

Having students generate diagrams is another possibility for helping them make sense of a numerical strategy (see **fig. 2**).

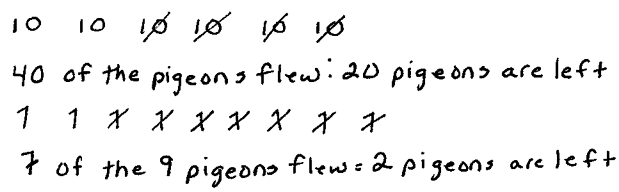
FIGURE 2

Two diagrams for Ian's solution strategy for 69 – 47 illustrate the reasoning behind his approach. Discussing similar diagrams and using the problem context helped students recognize that they could decompose numbers and take away pieces of numbers in chunks.

(a) Then they reasoned that composing twenty and two to find the solution made sense because that gives the number of pigeons left in the park.



(b) Creating and examining such diagrams help students make sense of the problem context and develop an understanding of the take-away meaning of subtraction (Schifter 2009).



Supplying a variety of solutions will create mathematical conflict and a basis for students to consider the problem's context and the solutions' reasonableness.

Besides highlighting the key steps in the solution methods when analyzing them, understanding a variety of solution methods is also important for students. For example, the following are two other common strategies for solving the same problem:

1. 69 – 47 = ?	2. 70 – 50 = 20
69 – 50 = 19	20 + 3 = 23
19 + 3 = 22	23 – 1 = 22

For the first method, it would be important to highlight that the problem was altered to having fifty pigeons fly away, rather than forty-seven; and because three too many flew away, they must be added back in. The second method is similar, but now one must highlight what changing sixty-nine to seventy means in the context of the problem (i.e., that there were supposed to be sixty-nine pigeons to begin with, so the extra one must be subtracted). Now consider the kinds of diagrams that students could generate to make sense of these two strategies and what they could learn by comparing them with the representations provided for the first strategy.

In summary, another way teachers can support reasoning is to have students analyze solution methods and provide justifications for the steps of these methods. The emphasis on reasoning behind solution methods helps students deepen their number sense and understanding of the meanings of the operations.

Efficiency

An additional issue that may influence the development of student reasoning, or lack thereof, is the extent to which a teacher highlights the efficiency and the sophistication of solution methods. A search for efficiency allows students to engage with new ways of thinking and recognize connections between the original problem and solutions that are more efficient. This last episode furnishes an example of how a teacher encouraged students to consider a more efficient solution. The specific instructional actions she takes—asking why more efficient solution methods work, using the context of the problems to restate questions about solution methods, and sending students back to small groups to reflect on shared ideas—are strategic. In this episode, fourth graders were presented with a seemingly straightforward computation problem:

> You have three decks of cards with fifty-two cards in each deck. How many cards will each person get if the three decks are dealt evenly to six people?

Having students solve problems in more than one way during individual work time usually allows them to consider different ways to think about problems. However, in this classroom, while students were sharing solutions in small groups, the teacher noticed that the majority of them were using only one strategy. They first found the total number of cards in three decks (156) and then divided by 6 to find the number of cards each person would receive. Only one student, Ryan, thought about simply dividing 52 by 2. When the teacher saw Ryan's work, she decided to create an opportunity for the whole group to consider his more efficient solution method. She initiated the discussion by summarizing the common strategy and then having students reason about Ryan's approach.

T: Ryan has on his paper fifty-two divided by two, and he proved that if you divide fifty in half, that's twenty-five; so fifty-two in half is twenty-six. Why do you suppose he is using those numbers instead of trying to figure out how many cards totally he has to deal out? [*pausing*] Mike, what do you think?
Mike: Not sure.
T: What kind of idea might Ryan be thinking of?

Kelly: I am not sure.
T: My question is, Do you have to figure out how many cards are in all three decks before you can actually solve the problem? You think so? What makes you say that?
Gabriela: Because, if you have fifty-two and you need to divide it by six [*pausing*]. . . .
T: Hmmmm. All right. Gail?
Gail: But I think what Ryan was doing is, since there are three decks of cards and you put two people to each deck [*pausing*] two people for each deck, and there are three decks. That would be six people for the total.
T: Who could repeat what Gail said? So, possibly they didn't hear you. Could you shout it out?
Gail: Well, Ryan put two people to each deck, since there are three decks, and since there are two people to each deck, then two times three equals six people.

Such questions as, "Why do you suppose he is using those numbers instead of trying to figure out how many cards totally he has to deal out?" have the potential to help students make sense of the new strategy as they stay connected to the context of the original problem. However, as it happened in the exchange above, students often struggle to understand right away what the reasoning is behind a different strategy. Rephrasing the original question, again using the context of the problem, could help students focus on the crux of the issue: Do you have to figure out how many cards are in all three decks before you can actually solve the problem? Ryan was solving the problem by thinking of six people sharing three decks of cards as two people sharing one deck of cards. By doing so, he transformed the original problem into an easier one to solve.

Although Gail was able to reason about Ryan's solution and shared his thinking twice, the rest of the group required more thinking time to understanding why $156 \div 6$ and $52 \div 2$ are equivalent problems. Once the teacher recognized their need, she sent the students back to small groups and asked them to compare the two strategies by generating diagrams. This gave the fourth graders the opportunity to focus on making sense of the relationship between the two solutions.

The exchange above suggests another technique teachers can use to promote reasoning: Encourage students to consider efficiency. Note that by using the context of the problem in her questioning, the teacher created an opportunity for students to make sense of the thinking behind the strategy. As students analyze solution methods for efficiency, they will learn how to compare the steps taken in the strategies and what these steps mean in terms of what the problem is asking. They will also identify similarities and differences among strategies, developing the understanding necessary to form connections among ideas embedded in these strategies.

Strategies with anticipated solutions

The kinds of tasks that children encounter in school influence the way they think about mathematics. The three tasks presented in this article were simple computation problems, but the teachers' orchestration of the discussions made the tasks mathematically challenging for their students. The goals of the discussions—to reflect on multiple solutions, analyze particular solution methods, and consider methods that are

Instructional strategies to promote understanding

Anticipate the solution methods that your students will use. Then strategically plan how you could use those methods to elicit students' thinking. To encourage students to make sense of the mathematics they are learning, teachers can give them the chance to mathematically disagree with one another, to reason about their perspectives, and to engage in conflict resolution. Key components of such classroom conversations might include the following specific instructional actions:

- To help students identify reasonable solutions, prompt them to consider whether more than one answer could be correct. Begin your whole-class discussions by providing all solutions, correct and incorrect, for students to contemplate.
- Help students make sense of numerical strategies by asking them to analyze solution methods. Cultivate the classroom expectation that they will provide justifications for the steps of these methods as well as counterexamples.
- Show your class how to represent word problems with diagrams and drawings (see **fig. 2**).
- When encouraging students to compare the efficiency of strategies, deliberately ask why the more efficient solution methods work. Use the problem context to restate your questions about solution methods. Have students meet in small groups to reflect on ideas that have been shared.

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Rich mathematical conversations help students develop deep understanding of context and the ability to reason flexibly.

more efficient—offer teachers leverage to focus children's reasoning. Instructional actions from these discussions—such as using the context of the problems when posing questions about solution methods, having students generate diagrams to justify their thinking, and asking why solution methods that are more efficient will work—were also key to keeping the focus on student thinking. These practices help create learning environments in which mathematical reasoning and sense making are crucial components of whole-group discussions. The message here is how important it is for you to think carefully about the solution methods you anticipate students using and to be strategic about the ways in which you will use those methods to support children's thinking.

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Nesrin Cengiz, nesrinc@umd.umich.edu, is an assistant professor of mathematics education at the University of Michigan–Dearborn. She is interested in how teachers help students learn mathematics with understanding through reasoning about mathematics.